The modified Ishikawa iteration process with errors in CAT(0) spaces

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Abstract

In this article, Δ-convergence and strong convergence of the modified Ishikawa iteration process with errors are established for continuous mappings of asymptotically nonexpansive type in CAT(0) spaces. Our results extend and improve the previous results given by many authors.

1 Introduction and Preliminaries

Let C be a closed convex subset of a metric space (X, d). A mapping $T : C \to C$ is said to be nonexpansive if and only if

$$d(Tx, Ty) \le d(x, y), \quad \forall x, y \in C.$$

A mapping $T : C \to C$ is called asymptotically nonexpansive if and only if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ for which

$$d(T^n x, T^n y) \le k_n d(x, y), \quad \text{for all } x, y \in C.$$

A mapping $T : C \to C$ is of asymptotically nonexpansive type if and only if T satisfies

$$\limsup_{n \to \infty} \sup_{y \in C} (d(T^n x, T^n y) - d(x, y)) \le 0 \quad \forall x \in C,$$

and T^n is continuous for some $n \geq 1$.

We denote the set of fixed points of T with $F(T)$; that is, $Fix(T) = \{x \in C : Tx = x\}$.

It is clear that nonexpansive and asymptotically nonexpansive mappings are mappings of asymptotically nonexpansive type.

Bruck et al. [3] introduced the notion of mappings which are asymptotically nonexpansive in the intermediate sense (continuous mappings of asymptotically nonexpansive type) and obtained the weak convergence theorems of averaging iteration for mappings of asymptotically nonexpansive type in the intermediate sense in uniformly convex Banach space with the Opial property. Since then many authors have studied existence and convergence theorems of fixed points for these two classes of mappings in Banach spaces, for example, Xu [23], Kaczor [15], Rhoades [20], etc.

Let C be a nonempty closed subset of a Hilbert space H and let $T : C \longrightarrow C$ be an asymptotically nonexpansive mapping. The modified Krasnoselski-Mann iteration process

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad \forall n \geq 1,$$

was studied by Schu [21] in Hilbert spaces, where $\{\alpha_n\} \subset [0,1]$ and $x_1 \in C$.

Tan and Xu [22] extended the result of Schu to uniformly convex Banach spaces for the modified Ishikawa iteration process

$$y_n = s_n T^n x_n + (1 - s_n)x_n,$$

$$x_{n+1} = t_n T^n y_n + (1 - t_n)x_n,$$

where $\{t_n\}$, $\{s_n\}$ are sequences in $[0,1]$ and $x_1 \in C$.

Now, we introduce CAT(0) spaces. Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x,y)] \longrightarrow X$ such that $c(0) = x, c(d(x,y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if and only if every two points of X are joined by a geodesic path, and X is said to be uniquely geodesic if and only if there is exactly one geodesic path joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining of any two its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices. A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ is the triangle $\overline{\triangle}(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j) \text{ for } i, j \in \{1, 2, 3\}.$$

A geodesic space X is a CAT(0) space if for each geodesic triangle $\triangle$ in X and its comparison triangle $\overline{\triangle}$ in $\mathbb{R}^2$, the CAT(0) inequality

$$d(x,y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y})$$

is satisfied for all $x, y \in \triangle$ and $\overline{x}, \overline{y} \in \overline{\triangle}$. The meaning of the CAT(0) inequality is that a geodesic triangle in X is at least thin as its comparison triangle in the Euclidean plane. It is well known that CAT(0) spaces are uniquely geodesic spaces.
There are several examples of CAT(0) spaces among metric structures. For example, any complete simply connected Riemannian manifold with nonpositive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces (see [1]), $\mathbb{R}$-trees (see [10]), Euclidean buildings (see [2]), complex Hilbert ball with a hyperbolic metric (see [8]). For a thorough discussion of these spaces and their fundamental role that play in geometry, we refer the reader to Bridson and Haefliger [1].
Fixed-point theory on CAT(0) spaces was first studied by Kirk (see [12, 13]). He showed that every nonexpansive (single-valued) mapping defined on a bounded, closed and convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multi-valued mappings in CAT(0) spaces has rapidly been developed. It is worth to mention that fixed-point theorems in CAT(0) spaces (specially in $\mathbb{R}$-trees) can be applied to graph theory, biology, and computer science.

In this article, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(z,x) = td(x,y), d(z,y) = (1-t)d(x,y)$. If X is a CAT(0) space and $x, y \in X$ then $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$ and subset C of X is convex if $[x,y] \subseteq C$, for all $x, y \in C$.

Lemma 1.1. *[6] Let (X,d) be a CAT(0) space. Then, for all $x, y, z \in X$ and all $t \in [0,1]$:*
(1) $d^2(tx \oplus (1-t)y, z) \le td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y),$
(2) $d(tx \oplus (1-t)y, z) \le td(x,z) + (1-t)d(y,z).$
In addition, by using (1), we have

$$d(tx \oplus (1-t)y, tx \oplus (1-t)z) \le (1-t)d(y,z).$$

We now give the concept of Δ-convergence and collect some of its basic properties.
Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subseteq X$ is given by

$$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subseteq X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}.$$

Recall that a bounded sequence $\{x_n\}$ in X is regular if and only if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

Lemma 1.2. *[5] If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space X and C is a closed and convex subset of X, then there exists a unique point $u \in C$ such that*

$$r(u, \{x_n\}) = \inf_{x \in C} d(x, x_n)$$

The following lemma is obtained by Lemma 1.2.

Lemma 1.3. *Let $\{x_n\}, C$ and X be as in Lemma 1.2. Then, $A(\{x_n\})$ and $A_C(\{x_n\})$ are singletons.*

Lemma 1.4. *[4] If C is a closed and convex subset of X and $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 1.5. *[6] If $\{x_n\}$ is a bounded sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lim [16] introduced the concept of "$\Delta - convergence$" in a general metric space. Kirk and Panyanak [14] specialized Lim's concept to CAT(0) spaces.

Definition 1.6. [14, 16] A sequence $\{x_n\}$ in X is Δ-convergent to $x \in X$ if and only if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - lim_{n \to \infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 1.7. *[14] Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Also, a notion of convergence in CAT(0) spaces was introduced by Hussain and Khamsi [9]. As follows,

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and C be a closed convex subset of X which contains $\{x_n\}$. We denote the notation

$$\{x_n\} \rightharpoonup w \text{ if and only if } \phi(w) = \inf_{x \in C} \phi(x)$$

where $\phi(x) = \limsup_{n \to \infty} d(x_n, x)$.

Lemma 1.8. *[24] Let X be a complete CAT(0) space and C be a nonempty, bounded, closed and convex subset of X. Then every mapping of asymptotically nonexpansive type $T : C \to C$ has a fixed point.*

Lemma 1.9. *[24] Let X be a complete CAT(0) metric space, C be a nonempty, bounded, closed and convex subset of X, $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type and $\{x_n\} \subset C$ be a sequence such that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ and $x_n \rightharpoonup w$. Then, $Tw = w$.*

The following Lemma shows the connection between " $\rightharpoonup$ " and Δ-convergence.

Lemma 1.10. *[17] Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and C be a closed and convex subset of X which contains $\{x_n\}$. Then,*
(1) $\Delta - \lim_{n \to \infty} x_n = x$ implies $x_n \rightharpoonup x$;
(2) if $\{x_n\}$ is regular, then $x_n \rightharpoonup x$ implies $\Delta - \lim_{n \to \infty} x_n = x$.

Easily, we can obtain the following corollary from Lemma 1.9 and Lemma 1.10 .

Corollary 1.11. *Let X be a complete CAT(0) metric space, C be a nonempty, bounded, closed and convex subset of X, $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type and $\{x_n\} \subset C$ be a sequence such that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n \to \infty} x_n = w$. Then, $Tw = w$.*

Lemma 1.12. *[24] Let X be a complete CAT(0) space, C be a closed and convex subset of X and $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type. Suppose that $\{x_n\}$ is a bounded sequence in C such that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ and $d(x_n, p)$ converges for each $p \in Fix(T)$, then $\omega_w(x_n) \subset Fix(T)$. Here $\omega_w(x_n) = \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Let C be a nonempty, closed and convex subset of a CAT(0) space X and $T : C \longrightarrow C$ be an asymptotically nonexpansive mapping. Δ-convergence of the modified Krasnoselski-Mann iteration process

$$x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n) x_n, \quad \forall n \geq 1, \tag{1.1}$$

was studied by Nanjaras and Panyanak [17] in CAT(0) spaces, where $\{\alpha_n\} \subset [0, 1]$ and $x_1 \in C$. Zhang and Cui [24] extended the results of Nanjaras and Panyanak [17] and obtained the existence of fixed points, demiclosed principle and Δ-convergence of the sequence generated by (1.1) for continuous mapping of asymptotically nonexpansive type. On the other hand, Panyanak and Laokul [19] introduced the modified Ishikawa iteration process in CAT(0) spaces X, as follow,

$$\begin{cases} y_n = s_n T x_n \oplus (1 - s_n) x_n, \\ x_{n+1} = t_n T y_n \oplus (1 - t_n) x_n, \end{cases} \tag{1.2}$$

where T is a nonexpansive mapping of $C \subset X$ to itself, $\{t_n\}$, $\{s_n\} \subset [0, 1]$ and $x_1 \in C$. They proved Δ-convergence of (1.2) under certain appropriate assumptions on $C \subset X$, $\{t_n\}$ and $\{s_n\}$.

In this article, we consider the modified Ishikawa iteration with errors for continuous mappings of asymptotically nonexpansive type in CAT(0) spaces and prove the strong convergence and Δ-convergence theorems of the sequence generated by the modified Ishikawa iteration with errors. Our results extend the results of Nanjaras and Panyanak [17], Zhang and Cui [24] and Panyanak and Laokul [19] and some other results in the literatures.

Lemma 1.13. *[18] Let $\{\alpha_n\}_{n \geq 1}$, $\{\beta_n\}_{n \geq 1}$ and $\{\gamma_n\}_{n \geq 1}$ be non-negative sequences satisfying*

$$\alpha_{n+1} \leq (1 + \gamma_n)\alpha_n + \beta_n, \ n \geq 1, \ \sum_{n=1}^{\infty} \gamma_n < \infty, \ \sum_{n=1}^{\infty} \beta_n < \infty.$$

Then $\lim \alpha_n$ exists. Moreover, if $\liminf_{n \to \infty} \alpha_n = 0$, then $\lim \alpha_n = 0$.

Lemma 1.14. *[17] Let X be a complete CAT(0) space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $\limsup_{n \to \infty} d(x_n, x) \leq r$, $\limsup_{n \to \infty} d(y_n, x) \leq r$, and $\lim_{n \to \infty} d((1 - t_n)x_n \oplus t_n y_n, x) = r$ for some $r \geq 0$. Then $\lim_{n \to \infty} d(x_n, y_n) = 0$.*

2 Main Results

In the following, we introduce the modified Ishikawa iteration process with errors for continuous mappings of asymptotically nonexpansive type at CAT(0) spaces setting and establish the strong convergence and Δ-convergence theorems of the sequence generated by the modified Ishikawa iteration with errors.

Definition 2.1. Let X be a complete CAT(0) space, C be a nonempty, bounded, closed and convex subset of X and $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type. Suppose that $x_1 \in C$ is arbitrary chosen, the modified Ishikawa iteration process $\{x_n\}$ with errors for continuous mappings of asymptotically nonexpansive type is iteratively generated by the following algorithm:

Step (1): Take $u_n = s_n T^n x_n \oplus (1 - s_n)x_n, \quad \forall n \geq 1$.

Step (2): Choose $y_n \in C$ such that $d(y_n, u_n) \leq a_n$, where $\{a_n\}$ is a sequence of nonnegative real numbers.

Step (3): Take $v_n = t_n T^n y_n \oplus (1 - t_n)x_n, \quad \forall n \geq 1$.

Step (4): Choose $x_{n+1} \in C$ such that $d(v_n, x_{n+1}) \leq b_n$, where $\{b_n\}$ is a sequence of nonnegative real numbers.

Step (5): Return to step (1).

Where $\{t_n\}$, $\{s_n\}$ are sequences in $[0, 1]$, $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$.

Lemma 2.2. *Let X be a complete CAT(0) space, C be a nonempty, bounded, closed and convex subset of X and $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type. Put*

$$c_n = \max\{0, \sup_{x,y \in C}\{d(T^n x, T^n y) - d(x, y)\}\}.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\{x_n\}$ is given by definition 2.1, then $\lim_{n \to \infty} d(x_n, p)$ exists for any $p \in Fix(T)$.

Proof. By using Lemma 1.8, let $p \in F(T)$. Then,

$d(x_{n+1}, p) \leq d(x_{n+1}, v_n) + d(v_n, p)$
$\leq b_n + t_n d(T^n y_n, p) + (1 - t_n)d(x_n, p)$
$\leq b_n + t_n[d(y_n, p) + c_n] + (1 - t_n)d(x_n, p)$
$\leq b_n + t_n d(y_n, u_n) + t_n d(u_n, p) + t_n c_n + (1 - t_n)d(x_n, p)$
$\leq b_n + t_n a_n + t_n s_n d(T^n x_n, p) + t_n(1 - s_n)d(x_n, p) + t_n c_n + (1 - t_n)d(x_n, p)$
$\leq b_n + t_n a_n + t_n s_n[d(x_n, p) + c_n] + t_n(1 - s_n)d(x_n, p) + t_n c_n + (1 - t_n)d(x_n, p)$
$= b_n + t_n a_n + t_n c_n + t_n s_n c_n + d(x_n, p)$
$\leq b_n + a_n + 2c_n + d(x_n, p),$

which by $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and Lemma 1.13 implies $\lim_{n \to \infty} d(x_n, p)$ exists. $\square$

Lemma 2.3. *Let X be a complete CAT(0) space, C be a nonempty, bounded, closed and convex subset of X and $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type. Put*

$$c_n = \max\{0, \sup_{x,y \in C}\{d(T^n x, T^n y) - d(x, y)\}\}.$$

Suppose $\sum_{n=1}^{\infty} c_n < \infty$, $\{x_n\}$ is given by definition 2.1 such that $\{t_n\}$ is in $[a, b]$ for some $a, b \in (0, 1)$ and $\limsup_{n \to \infty} s_n < 1$. Then,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Proof. By Lemma 1.8, let $p \in F(T)$. It follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p) = r$. We have
$$d(T^n y_n, p) \leq d(y_n, p) + c_n \leq d(y_n, u_n) + d(u_n, p) + c_n$$

$$\leq a_n + s_n d(T^n x_n, p) + (1 - s_n) d(x_n, p) + c_n$$

$$\leq a_n + s_n d(x_n, p) + s_n c_n + (1 - s_n) d(x_n, p) + c_n$$

$$= a_n + c_n + s_n c_n + d(x_n, p).$$

$$\leq a_n + 2c_n + d(x_n, p).$$

By $\lim_{n\to\infty}(a_n + 2c_n) = 0$, we obtain,

$$\limsup_{n\to\infty} d(T^n y_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = r.$$

On the other hand, $|d(x_{n+1}, p) - d(v_n, p)| \leq b_n$ and $b_n \to 0$ imply

$$\lim_{n\to\infty} d(t_n T^n y_n \oplus (1 - t_n)x_n, p) = \lim_{n\to\infty} d(v_n, p) = \lim_{n\to\infty} d(x_{n+1}, p) = r.$$

Hence, by Lemma 1.14, $\lim_{n\to\infty} d(x_n, T^n y_n) = 0$. Moreover,
$$d(x_n, T^n x_n) \leq d(x_n, T^n y_n) + d(T^n y_n, T^n x_n)$$

$$\leq d(x_n, T^n y_n) + d(y_n, x_n) + c_n$$

$$\leq d(x_n, T^n y_n) + d(y_n, u_n) + d(u_n, x_n) + c_n$$

$$\leq d(x_n, T^n y_n) + a_n + s_n d(T^n x_n, x_n) + c_n.$$
Therefore,
$$d(x_n, T^n x_n) \leq \frac{d(x_n, T^n y_n) + a_n + c_n}{1 - s_n},$$

which by $\limsup_n s_n < 1$, $a_n \to 0$, $c_n \to 0$ and $d(x_n, T^n y_n) \to 0$, implies

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0.$$

In addition,
$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n)$$

$$\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(x_{n+1}, x_n) + c_{n+1} + d(T^{n+1} x_n, Tx_n)$$

$$= 2d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + c_{n+1} + d(T^{n+1}x_n, Tx_n)$$

$$\leq 2d(x_n, v_n) + 2d(v_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + c_{n+1} + d(T^{n+1}x_n, Tx_n)$$

$$\leq 2t_n d(x_n, T^n y_n) + 2b_n + d(x_{n+1}, T^{n+1}x_{n+1}) + c_{n+1} + d(T(T^n x_n), Tx_n).$$

Hence, by $b_n \to 0$, $c_n \to 0$, $d(x_n, T^n y_n) \to 0$, $d(x_n, T^n x_n) \to 0$ and the uniform continuity of T, we conclude

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0.$$

$\square$

Theorem 2.4. *Let X be a complete CAT(0) space, C be a nonempty, bounded, closed and convex subset of X and $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type. Put*

$$c_n = \max\{0, \sup_{x,y \in C}\{d(T^n x, T^n y) - d(x, y)\}\}.$$

Suppose $\sum_{n=1}^{\infty} c_n < \infty$, $\{x_n\}$ is given by definition 2.1 such that $\{t_n\}$ is in $[a, b]$ for some $a, b \in (0, 1)$, and $\limsup_{n \to \infty} s_n < 1$. Then $\{x_n\}$ is Δ-convergent to a fixed point of T.

Proof. By Lemma 2.2, $\{d(x_n, p)\}$ is convergent for all $p \in F(T)$ and Lemma 2.3 implies $\lim_{n \to \infty} d(x_n, Tx_n) = 0$. Therefore, by Lemma 1.12, $\omega_w(x_n)$ consists exactly one point and is a subset of $F(T)$. This shows that $\{x_n\}$ is Δ-convergent to an element of $F(T)$. $\square$

Theorem 2.5. *Let X be a complete CAT(0) space, C be a nonempty compact convex subset of X and $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type. Put*

$$c_n = \max\{0, \sup_{x,y \in C}\{d(T^n x, T^n y) - d(x, y)\}\}.$$

If $\sum_{n=1}^{\infty} c_n < \infty$, $\{x_n\}$ is given by definition 2.1 such that $\{t_n\}$ is in $[a, b]$ for some $a, b \in (0, 1)$, and $\limsup_{n \to \infty} s_n < 1$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By compactness of C, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly to an element in C. Let $\lim_k x_{n_k} = z$, then

$$d(z, Tz) \leq d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tz) \leq 2d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}).$$

By Lemma 2.3, $d(x_{n_k}, Tx_{n_k}) \to 0$ and $\lim_k x_{n_k} = z$, we obtain

$$d(z, Tz) = 0,$$

which implies $z \in F(T)$. Therefore, Lemma 2.2 and $\lim_k d(x_{n_k}, z) = 0$ imply

$$\lim_n d(x_n, z) = 0.$$

Hence, $\{x_n\}$ converges strongly to $z \in F(T)$. $\qquad\square$

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